

# Multiparty multilevel Greenberger-Horne-Zeilinger states

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The proof of Bell's theorem without inequalities by Greenberger, Horne, and Zeilinger (GHZ) is extended to multipartite multilevel systems. The proposed procedure generalizes previous partial results and provides an operational characterization of the so-called GHZ states for multipartite multilevel systems.

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## I. INTRODUCTION

Greenberger, Horne, and Zeilinger (GHZ) [1,2] show that the quantum predictions for an individual system composed of three or more particles prepared in a specific entangled state (henceforth called a “GHZ state”) cannot be reproduced by any local hidden-variables model based on the definition of “elements of reality” proposed by Einstein, Podolsky, and Rosen (EPR) [3]. The proof was originally developed for four spin- $\frac{1}{2}$  particles [1], later simplified to three spin- $\frac{1}{2}$  particles [2,4], and has been recently verified experimentally [5,6]. On the other hand, the GHZ proof has been extended to  $n$  spin- $\frac{1}{2}$  particles [7,8], and to three *pairs* of spin- $\frac{1}{2}$  particles [9]. In addition, GHZ-like proofs for particular states of  $n$  spin- $s$  particles have been investigated [10]. So far, however, no generalization of the GHZ proof for multipartite multilevel systems has been considered. This extension would be of interest since it would lead to a definition of GHZ states for multipartite multilevel systems. Particularly because the classification of the pure states of single copies of multipartite multilevel systems becomes highly complicated beyond three qubits [11,12].

The structure of this paper is as follows: in Sec. II, a GHZ-like proof for three three-level systems is proposed. In Sec. III, the GHZ proof is extended to three  $m$ -level systems. Additionally, in Sec. IV, a proof of the Kochen-Specker theorem [13] based on the GHZ-like proof for the case of  $m$  being an even number is presented. In Sec. V, the GHZ proof is generalized to  $n$  particles of  $m$  levels, and to  $n$  particles with varying number of levels (all with the same parity). The main point of this paper is not the generalization itself but to illustrate the basic ingredients of *any* GHZ-like proof, so that we can provide a natural definition of what could be called a “GHZ state” in the

context of multipartite multilevel systems. This definition is presented in Sec. VI.

## II. GHZ-LIKE PROOF FOR THREE THREE-LEVEL SYSTEMS

The common scenario to any GHZ-like proof is the following: A system composed of three (or  $N \geq 3$ ) particles is initially prepared in a specific pure entangled state (a GHZ state). Each particle moves away to a distant space-time region where an observer measures either  $A_i$  or  $B_i$ , where  $i$  denotes particle  $i$ . Local measurements on particle  $i$  are assumed to be spacelike separated from local measurements on the other particles. For certain combinations of measurements, if all the observers except observer  $N$  share their results, then they can predict with certainty the result of measuring  $A_N$  (or  $B_N$ ) on particle  $N$ . Therefore, adopting the EPR criterion of elements of reality [3], there must be an element of reality (a value) corresponding to  $A_N$  ( $B_N$ ). Similar reasonings lead us to conclude that all the one-particle observables  $A_i$  and  $B_i$  must have predefined values. The proof concludes by showing that one cannot assign values to all these one-particle observables in a way consistent with all the quantum predictions.

In order to construct a GHZ proof for the case of three three-level subsystems (for instance, three spin-1 particles), we must look for two one-particle maximal operators for each quantum three-level subsystem (or “qutrit”)  $i$ ,  $A_i$  and  $B_i$ , such that they anticommute, that is

$$A_i B_i = -B_i A_i. \quad (1)$$

For instance [14],

$$A_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad (2)$$

$$B_i = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad (3)$$

Then, automatically, the four operators defined as

$$A_1 B_2 B_3 = A_1 \otimes B_2 \otimes B_3, \quad (4)$$

$$B_1 A_2 B_3 = B_1 \otimes A_2 \otimes B_3, \quad (5)$$

$$B_1 B_2 A_3 = B_1 \otimes B_2 \otimes A_3, \quad (6)$$

$$A_1 A_2 A_3 = A_1 \otimes A_2 \otimes A_3. \quad (7)$$

are mutually commutative, and therefore possess a set of common eigenvectors. The eigenvalues of these four

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operators are  $-1$  (four of them),  $0$  (nineteen of them), and  $1$  (four of them). In addition, the product

$$C = A_1 B_2 B_3 \times B_1 A_2 B_3 \times B_1 B_2 A_3 \times A_1 A_2 A_3 \quad (8)$$

is a negative operator; its eigenvalues are either  $-1$  (eight of them) or  $0$  (nineteen). Suppose we chose a common eigenvector  $|\mu\rangle$  of the four operators such that

$$A_1 B_2 B_3 |\mu\rangle = |\mu\rangle, \quad (9)$$

$$B_1 A_2 B_3 |\mu\rangle = |\mu\rangle, \quad (10)$$

$$B_1 B_2 A_3 |\mu\rangle = |\mu\rangle, \quad (11)$$

$$A_1 A_2 A_3 |\mu\rangle = -|\mu\rangle. \quad (12)$$

This eigenvector is

$$|\mu\rangle = \frac{1}{2} (|11\bar{1}\rangle + |1\bar{1}1\rangle + |\bar{1}11\rangle - |\bar{1}\bar{1}\bar{1}\rangle), \quad (13)$$

where

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |\bar{1}\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (14)$$

Now consider three observers, each having access to one particle. On particle  $i$  the corresponding observer measures either  $A_i$  or  $B_i$  without disturbing the other particles. The results of these measurements will be called  $a_i$  or  $b_i$ , respectively. Since these results must satisfy the same functional relations satisfied by the corresponding operator, then, from Eq. (9), we can predict that, if  $A_1$ ,  $B_2$ , and  $B_3$  are measured, their results must satisfy

$$a_1 b_2 b_3 = 1. \quad (15)$$

Analogously, from Eqs. (10)-(12), the results of other possible measurements must satisfy

$$b_1 a_2 b_3 = 1, \quad (16)$$

$$b_1 b_2 a_3 = 1, \quad (17)$$

$$a_1 a_2 a_3 = -1. \quad (18)$$

We can associate each of the eigenvalues  $a_i$  and  $b_i$  to an EPR element of reality [3] of particle  $i$ , initially hidden in the original state of the system, but “revealed” by performing measurements on the other two distant particles. For example, if the observers on particles 1 and 2 measure, respectively,  $A_1$  and  $B_2$ , and their results are both 1, then, sharing their results and using Eq. (15), they can predict with certainty that the result of measuring  $B_3$  will be 1. Since arriving to this conclusion does not require any real interaction with particle 3, then, according to EPR, particle 3 has the value 1 for  $B_3$ , so we can assign the value 1 to the observable  $B_3$ . Alternatively, since a different measurement on particles 1 and 2 (for instance, by measuring  $B_1$  instead of  $A_1$ ) allows the observers of particles 1 and 2 to predict with certainty, and without interacting with particle 3, the result of  $A_3$  — using Eq. (17)—, then we suppose that this result was

somehow predetermined. Such predictions with certainty would lead us to assign values to the six observables  $A_1$ ,  $B_1$ ,  $A_2$ ,  $B_2$ ,  $A_3$ , and  $B_3$ . However, such assignment cannot be consistent with the rules of quantum mechanics because the four equations (15)-(18) cannot be satisfied simultaneously, since the product of their left-hand sides is a positive number (because each value appears twice), whereas the product of the right-hand sides is  $-1$ . Therefore, the values of these observables cannot be predefined as we assumed. Note that we can also develop a similar reasoning if we choose any other common eigenvector of the four operators so that the product of the corresponding eigenvalues is negative.

### III. GHZ-LIKE PROOF FOR THREE $M$ -LEVEL SYSTEMS

The method used in the previous section can be easily extended to any system composed of three spin- $s$  (or  $m$ -level, with  $m = 2s + 1$ ) systems. We will distinguish between the case of  $m$  being an odd number and the case of  $m$  being an even number.

If  $m$  is an odd number, we can choose the following one-particle anticommutative operators for each subsystem  $i$ :

$$A_i = \begin{pmatrix} s & & & & & & \\ & s-1 & & & & & \\ & & \dots & & & & \\ & & & 1 & & & \\ & & & & 0 & & \\ & & & & & -1 & \\ & & & & & & \dots \\ & & & & & & & -s+1 \\ & & & & & & & & -s \end{pmatrix}, \quad (19)$$

$$B_i = \begin{pmatrix} & & & & & & s \\ & & & & & & & s-1 \\ & & & & & & & & \dots \\ & & & & & & & & & 1 \\ & & & & & & & & & & 0 \\ & & & & & & & & & & & 1 \\ & & & & & & & & & & & & \dots \\ & & & & & & & & & & & & & s-1 \\ s & & & & & & & & & & & & & \end{pmatrix}. \quad (20)$$

The other entries in the matrices are assumed to be zeroes. The argument of nonlocality is almost identical to that in Sec. II. The only difference is that in this case the eigenvalues of the four operators (4)-(7) are positive ( $[(2s+1)^3 - k]/2$  of them, being  $k = 12s^2 + 6s + 1$ ), zero ( $k$  of them), and negative ( $[(2s+1)^3 - k]/2$  of them), and the eigenvalues of product (8) are negative ( $[2s+1]^3 - k$  of them) and zero ( $k$  of them).

If  $m$  is an even number, we can choose the following one-particle anticommutative operators:

$$A_i = \begin{pmatrix} s & & & & & \\ & s-1 & & & & \\ & & \dots & & & \\ & & & 1 & & \\ & & & & -1 & \\ & & & & & \dots \\ & & & & & & -s+1 \\ & & & & & & & -s \end{pmatrix}, \quad (21)$$

$$B_i = \begin{pmatrix} & & & & & s \\ & & & & s-1 & \\ & & & 1 & \dots & \\ & & 1 & & & \\ & & & & & \\ & s-1 & \dots & & & \\ s & & & & & \end{pmatrix}, \quad (22)$$

and develop a similar argument. The big difference in the case in which  $m$  is an even number is that the one-particle operators (21) and (22) have no zero eigenvalues, so the product of the four operators (8) is a definite negative operator (i.e., all its eigenvalues are negative). Thus, *every* common eigenvector of the four operators will allow us to develop a GHZ-like argument.

#### IV. KOCHEN-SPECKER PROOF FOR THREE $M$ -LEVEL SYSTEMS, WITH $M$ BEING AN EVEN NUMBER

Indeed, the last result of the previous section allows us to develop a “multiplicative” proof of the Kochen-Specker theorem [13] for a three  $m$ -level system, being  $m$  an even number, that generalizes those multiplicative proofs proposed by Mermin for three spin- $\frac{1}{2}$  particles [15] or by Cabello for three pairs of spin- $\frac{1}{2}$  particles [9]. As seen above, in the case of  $m$  being an even number, a GHZ-like proof could be developed starting from any common eigenvector of the four operators (4)-(7). Therefore, the argument can be rearranged as a state-independent proof of the Kochen-Specker theorem in an  $m^3$ -dimensional Hilbert space (with  $m$  being an even number) just with the inclusion of these four operators. The resulting proof of the Kochen-Specker theorem is summarized in Fig. 1, which contains ten operators: the four operators (4)-(7) acting on the whole system, and the six one-particle operators  $A_i$  and  $B_i$ . The four operators on each of the five straight lines are mutually commutative. As stated above, the product of the four operators on the horizontal line is a definite negative operator and, as can be easily verified, the product of the four operators on each of the other lines is one and the same definite positive operator. It can be easily checked that it is impossible to ascribe one of their eigenvalues to each of the ten operators, satisfying the same functional relations that are satisfied by the corresponding operators.

#### V. GHZ-LIKE PROOF FOR $N$ $M$ -LEVEL SYSTEMS

Let us extend the GHZ proof to the case of a system with  $n$  subsystems, all of them with  $m$  levels. It is convenient to distinguish between the case in which  $n$  is an odd number and the case in which  $n$  is an even number.

In case of  $n$  being an odd number, the proofs (for  $m$  being an odd number, or for  $m$  being an even number) will be similar to the proofs in Sec. III: We will use the same one-particle operators  $A_i$  and  $B_i$  (now with  $i$  from 1 to  $n$ ), and we will construct four  $n$ -particle operators of the form  $O_1 \otimes O_2 \otimes \dots \otimes O_n$ , where  $O_i$  is  $A_i$  or  $B_i$ . These  $n$ -particle operators must satisfy the following requirements [16]: (i) In order to commute, the  $n$ -particle operators must contain a number of operators of the  $A_i$  kind with the same parity in all of them (thus the parity of the operators  $B_i$  will also be the same). (ii) In order to avoid the product of the four  $n$ -particle operators having positive eigenvalues, one of the  $n$ -particle operators must have a different number (but with the same parity) of operators of the kind  $A_i$  than the other three. (iii) In order to obtain a GHZ-like algebraic (parity) contradiction, each one-particle operator must be used in the construction of two  $n$ -particle operators. (iv) In order to obtain a nontrivial proof (in the sense that all particles are required for the contradiction), all one-particle operators must be used in the definition of the  $n$ -particle operators. For instance, for  $n = 5$ , the following four operators allow us to develop a GHZ-like proof in a similar way as in Sec. III:

$$A_1 \otimes B_2 \otimes B_3 \otimes B_4 \otimes B_5, \quad (23)$$

$$A_1 \otimes A_2 \otimes A_3 \otimes B_4 \otimes B_5, \quad (24)$$

$$B_1 \otimes B_2 \otimes A_3 \otimes A_4 \otimes A_5, \quad (25)$$

$$B_1 \otimes A_2 \otimes B_3 \otimes A_4 \otimes A_5. \quad (26)$$

The one-particle operators  $A_i$  and  $B_i$  can be, respectively, those of Eqs. (19) and (20), if  $m$  is an odd number, or those of Eqs. (21) and (22), if  $m$  is an even number.

The case of  $n$  particles with  $n$  being an even number is more complicated since, as can be easily checked, the requirements (i)-(iv) cannot be satisfied. A simple trick for developing a proof in this case is as follows: Start with a GHZ proof for  $n - 1$  particles, then construct four  $n$ -particle operators just by adding the same one-particle operator to the four  $(n - 1)$ -particle operators (i.e., by making their tensor product with the same one-particle operator). For instance, in order to construct a proof for  $n = 4$ , let us start with the four three-particle operators given by Eqs. (4)-(7), and add the one-particle operator  $B_4$ . This leads to the following four-particle operators:

$$A_1 \otimes B_2 \otimes B_3 \otimes B_4, \quad (27)$$

$$B_1 \otimes A_2 \otimes B_3 \otimes B_4, \quad (28)$$

$$B_1 \otimes B_2 \otimes A_3 \otimes B_4, \quad (29)$$

$$A_1 \otimes A_2 \otimes A_3 \otimes B_4. \quad (30)$$

Note that this set of operators does not satisfy (iv). In order to fulfill (iv), we can add one new four-particle operator ending on  $A_4$  and containing a number of  $A_i$  and  $B_i$  with the same parity as the operators (27)-(30). For instance,

$$B_1 \otimes B_2 \otimes B_3 \otimes A_4. \quad (31)$$

Note that then the product of the operators (27)-(31) would contain positive eigenvalues. However, in order to avoid this, we can consider the product of six operators, being the new one the same defined in (31). The rest of the proof is as in Sec. III.

The same method can be applied to any system composed of  $n$  parts with, respectively,  $m_1, m_2, \dots, m_n$  levels. The only restriction is that  $m_1, m_2, \dots, m_n$  must have the same parity.

On the other hand, if all the  $n$ -particle operators have the same eigenvalues and none of them is zero, then one can develop a Kochen-Specker multiplicative proof for the corresponding Hilbert space in a similar way to that of Sec. IV.

## VI. WHAT IS A GHZ STATE?

The aim of this work is not only to show how to develop a GHZ-like proof for multiparty multilevel quantum systems, but also to illustrate the basic ingredients of any GHZ-like proof, and provide a natural definition of GHZ states in this context. This is of interest since the classification of pure states of single copies of composite systems becomes more difficult as we increase the number of parts or the number of levels. In fact, this classification is highly difficult beyond three-particle two-level systems [11,12]. Therefore, the question of what is a GHZ state for a multiparty multilevel quantum system is not trivial. According to the proofs presented in this paper, a natural definition of GHZ states for multiparticle multilevel systems is any for which: (I) for every subsystem there are two one-particle (maximal or not) anticommutative operators  $A_i$  and  $B_i$ , such that (II) the state of the system is an eigenvector of a set of four (in the case of subsystems with an even number of levels) or five (in the case of subsystems with an odd number of levels)  $n$ -particle operators constructed as a tensor product of these one-particle operators, and such that (III) the product of the corresponding eigenvalues leads to an algebraic (parity) contradiction if one assumes EPR elements of reality. For instance, these criteria allow us to say that the three two-level state, given by [12]

$$|W\rangle = \frac{1}{\sqrt{3}} (|001\rangle + |010\rangle + |100\rangle), \quad (32)$$

where

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (33)$$

is not a GHZ state, since it is not a common eigenvector of four commuting operators of the form  $O_1 \otimes O_2 \otimes O_3$ , although it is a “maximally entangled” state in the sense described in [12]. In addition, the proofs presented in this work provide a constructive method to generate GHZ states for any multiparty multilevel (all with the same parity) system.

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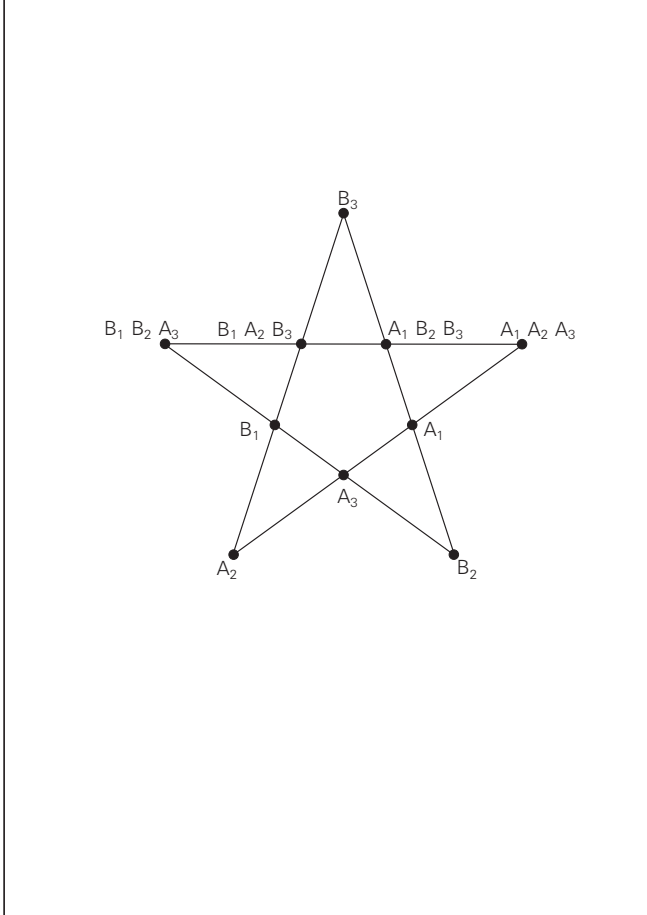


FIG. 1: Each dot represents an observable. The ten observables provide a proof of the Kochen-Specker theorem in a Hilbert space of dimension  $m^3$ , with  $m$  being an even number. The four observables on each line are mutually compatible and the product of their results must be positive, except for the horizontal line, where the product must be negative.